4^{th} power

The values are V_{008}, V_{017}, V_{026}, V_{035}, V_{044}, V_{116}, V_{125}, V_{134}, V_{224}, and V_{233}. The variables α_{ijk} come from the second power: $\alpha_{004}, \alpha_{013}, \alpha_{022}, \alpha_{112}$.

$$X_0 = 2\alpha_{002} + \alpha_{011} \qquad X_1 = 2\alpha_{011} \qquad X_2 = \alpha_{002} \tag{1}$$
$$Y_0 = 2\alpha_{002} + \alpha_{011} \qquad Y_1 = 2\alpha_{011} \qquad Y_2 = \alpha_{002} \tag{2}$$
$$Z_0 = 2\alpha_{002} + \alpha_{011} \qquad Z_1 = 2\alpha_{011} \qquad Z_2 = \alpha_{002} \tag{3}$$

The system has rank $2+2+1-2 = 3$ (or maybe $3+3+3-2 = 7$? but there are clearly many linearly dependent equations here) and only 2 variables, so $\Delta = \emptyset$, and we can solve it directly:

$$\alpha_{002} = X_2 = Y_2 = Z_2 = X_0/2 - X_1/4 = Y_0/2 - Y_1/4 = Z_0/2 - Z_1/4 \tag{4}$$
$$\alpha_{011} = X_1/2 = Y_1/2 = Z_1/2 = X_0 - 2X_2 = Y_0 - 2Y_2 = Z_0 - 2Z_2 \tag{5}$$

or in other words,

$$\frac{\partial(\alpha_{002}, \alpha_{011})}{\partial(X_0, X_1, X_2, Y_0, Y_1, Y_2, Z_0, Z_1, Z_2)} = \begin{bmatrix} 1/2 & -1/4 & 1 & 1/2 & -1/4 & 1 & 1/2 & -1/4 & 1 \\ 1 & 1/2 & -2 & 1 & 1/2 & -2 & 1 & 1/2 & -2 \end{bmatrix} \tag{6}$$

V_{004}.

The only base is $W_{002} = V_{002}V_{002} = 1$. ($W_{011}$ doesn't exist because then j would be negative in $V_{0,-1,1}$—right? I'm calling W_{ijk} "bases," since they become the exponential bases in the nx, *etc.* calculations—if you have your own names for these assorted variables, let me know and I'll change the names in my program.)

The products $(nx_\ell, ny_\ell, nz_\ell, n\alpha_{ijk})$ are

$$nx_0 = W_{002}^{3\partial\alpha_{002}/\partial X_0} = 1^{3\cdot 1/2} = 1 \qquad nx_1 = 1^{3\cdot -1/4} = 1 \qquad nx_2 = 1^{3\cdot 1} = 1 \tag{7}$$
$$ny_0 = 1^{3\cdot 1/2} = 1 \qquad ny_1 = 1^{3\cdot -1/4} = 1 \qquad ny_2 = 1^{3\cdot 1} = 1 \tag{8}$$
$$nz_0 = 1^{3\cdot 1/2} = 1 \qquad nz_1 = 1^{3\cdot -1/4} = 1 \qquad nz_2 = 1^{3\cdot 1} = 1 \tag{9}$$

There are no $n\alpha_{ijk}$, since Δ is empty, so their product is trivially 1 and we have

$$V_{004} = (3)^{1/3}(3)^{1/3}(3)^{1/3} = \boxed{3} \qquad \text{—should be } 1\,. \tag{10}$$

V_{013}.

The bases are

$$W_{002} = V_{002}V_{011} = 1 \cdot q^\tau = q^\tau \tag{11}$$
$$W_{011} = V_{011}V_{002} = q^\tau \tag{12}$$

The products are

$$nx_0 = W_{002}^{3\partial\alpha_{002}/\partial X_0} W_{011}^{3\partial\alpha_{011}/\partial X_0} = (q^\tau)^{3\cdot 1}(q^\tau)^{3\cdot 1} = q^{6\tau} \tag{13}$$
$$ny_0 = (q^\tau)^{3\cdot 1}(q^\tau)^{3\cdot -1} = 1 \tag{14}$$
$$ny_1 = (q^\tau)^{3\cdot -1}(q^\tau)^{3\cdot 1} = 1 \tag{15}$$
$$nz_1 = (q^\tau)^{3\cdot -1}(q^\tau)^{3\cdot 1} = 1 \tag{16}$$
$$nz_2 = (q^\tau)^{3\cdot 1}(q^\tau)^{3\cdot -1} = 1 \tag{17}$$

so

$$V_{013} = (q^{6\tau})^{1/3}(1+1)^{1/3}(1+1)^{1/3} = \boxed{2^{2/3}q^{2\tau}} \qquad \text{—should be } (2q)^\tau\,. \tag{18}$$

V_{022}.

The bases are

$$W_{002} = V_{002}V_{020} = V_{002}^2 = 1 \tag{19}$$
$$W_{011} = V_{011}V_{011} = (q^\tau)^2 = q^{2\tau} \tag{20}$$

The products are

$$nx_0 = W_{002}^{3\partial\alpha_{002}/\partial X_0} W_{011}^{3\partial\alpha_{011}/\partial X_0} = 1^{3\cdot 1}(q^{2\tau})^{3\cdot 1} = q^{6\tau} \tag{21}$$
$$ny_0 = 1^{3\cdot 1}(q^{2\tau})^{3\cdot -1} = q^{-6\tau} \tag{22}$$
$$ny_1 = 1^{3\cdot -1}(q^{2\tau})^{3\cdot 1} = q^{6\tau} \tag{23}$$
$$nz_1 = 1^{3\cdot -1}(q^{2\tau})^{3\cdot 1} = q^{6\tau} \tag{24}$$
$$nz_2 = 1^{3\cdot 1}(q^{2\tau})^{3\cdot -1} = q^{-6\tau} \tag{25}$$

so

$$V_{022} = (q^{6\tau})^{1/3}(q^{-6\tau} + q^{6\tau})^{1/3}(q^{6\tau} + q^{-6\tau})^{1/3} \tag{26}$$
$$= \boxed{(1+q^{12\tau})^{1/3}(q^{6\tau} + q^{-6\tau})^{1/3}} \qquad \text{—should be } (q^2+2)^\tau\,. \tag{27}$$

V_{112}.

The bases are

$$W_{002} = V_{002}V_{110} = V_{002}V_{011} = q^{\tau} \tag{28}$$
$$W_{011} = V_{011}V_{101} = V_{011}^2 = q^{2\tau} \tag{29}$$

The products are

$$nx_0 = W_{002}^{3\partial\alpha_{002}/\partial X_0} W_{011}^{3\partial\alpha_{011}/\partial X_0} = (q^{\tau})^{3\cdot 1}(q^{2\tau})^{3\cdot 1} = q^{9\tau} \tag{30}$$
$$ny_0 = (q^{\tau})^{3\cdot 1}(q^{2\tau})^{3\cdot -1} = q^{-3\tau} \tag{31}$$
$$ny_1 = (q^{\tau})^{3\cdot -1}(q^{2\tau})^{3\cdot 1} = q^{3\tau} \tag{32}$$
$$nz_1 = (q^{\tau})^{3\cdot -1}(q^{2\tau})^{3\cdot 1} = q^{3\tau} \tag{33}$$
$$nz_2 = (q^{\tau})^{3\cdot 1}(q^{2\tau})^{3\cdot -1} = q^{-3\tau} \tag{34}$$

so

$$V_{022} = (q^{9\tau})^{1/3}(q^{-3\tau} + q^{3\tau})^{1/3}(q^{3\tau} + q^{-3\tau})^{1/3} \tag{35}$$
$$= \boxed{q^{\tau}(1 + q^{6\tau})^{2/3}} \qquad \text{—should be } 2^{2/3}q^{\tau}(q^{3\tau} + 2)^{1/3}\,. \tag{36}$$