2^{nd} power

The values are V_{004}, V_{013}, V_{022}, V_{112}. The variables α_{ijk} have to come from the first power: $\alpha_{002}, \alpha_{011}, \alpha_{101}, \alpha_{110}$. (Why 110 and 101, but not 200 or 020?)

$$X_0 = \alpha_{002} + \alpha_{011} \qquad X_1 = \alpha_{101} + \alpha_{110} \tag{1}$$
$$Y_0 = \alpha_{002} + \alpha_{101} \qquad Y_1 = \alpha_{011} + \alpha_{110} \tag{2}$$
$$Z_0 = \alpha_{110} \qquad Z_1 = \alpha_{011} + \alpha_{101} \qquad Z_2 = \alpha_{002} \tag{3}$$

The system has rank $2+2+3-3 = 4$ and 4 variables, so $\Delta = \emptyset$, and we can solve it directly:

$$\alpha_{002} = Z_2 \tag{4}$$
$$\alpha_{011} = X_0 - Z_2 \tag{5}$$
$$\alpha_{101} = Z_1 - X_0 + Z_2 \tag{6}$$
$$\alpha_{110} = Z_0 \tag{7}$$

or in other words,

$$\frac{\partial(\alpha_{002}, \alpha_{011}, \alpha_{101}, \alpha_{110})}{\partial(X_0, X_1, Y_0, Y_1, Z_0, Z_1, Z_2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{8}$$

(Are those zeroes really zeroes? What if I solve the equations in a different way?)

V_{112}.

The bases W_{ijk} are

$$W_{002} = V_{002}V_{110} = q^\tau \cdot 1 = q^\tau \tag{9}$$
$$W_{011} = V_{011}V_{101} = (q^\tau)^2 = q^{2\tau} \tag{10}$$
$$W_{101} = V_{101}V_{011} = (q^\tau)^2 = q^{2\tau} \tag{11}$$
$$W_{110} = V_{110}V_{002} = q^\tau \cdot 1 = q^\tau \tag{12}$$

The products $(nx_\ell, ny_\ell, nz_\ell, n\alpha_{ijk})$ are

$$nx_0 = W_{011}^{3\partial\alpha_{011}/\partial X_0} W_{101}^{3\partial\alpha_{101}/\partial X_0} = W_{011}^3 W_{101}^{-3} = 1 \tag{13}$$
$$nx_1 = 1 \tag{14}$$
$$ny_0 = 1 \tag{15}$$
$$ny_1 = 1 \tag{16}$$
$$nz_0 = W_{110}^3 = q^{3\tau} \tag{17}$$
$$nz_1 = W_{101}^3 = q^{6\tau} \tag{18}$$
$$nz_2 = W_{002}^3 W_{011}^{-3} W_{101}^3 = q^{3\tau} \tag{19}$$

There are no $n\alpha_{ijk}$, since Δ is empty, so their product is trivially 1 and we have

$$V_{022} = (1+1)^{1/3}(1+1)^{1/3}(q^{3\tau} + q^{6\tau} + q^{3\tau})^{1/3} \tag{20}$$
$$= 2^{2/3}(q^{3\tau})^{1/3}(2 + q^{3\tau})^{1/3} \qquad \text{—should be } 2^{2/3}q^\tau(q^{3\tau}+2)^{1/3} \checkmark \tag{21}$$

2^{nd} power

The same system above can be solved as:

$$\alpha_{002} = Z_2 \tag{22}$$

$$\alpha_{011} = Y_1 - Z_0 \tag{23}$$

$$\alpha_{101} = -Y_1 + Z_0 + Z_1 \tag{24}$$

$$\alpha_{110} = Z_0 \tag{25}$$

or in other words,

$$\frac{\partial(\alpha_{002}, \alpha_{011}, \alpha_{101}, \alpha_{110})}{\partial(X_0, X_1, Y_0, Y_1, Z_0, Z_1, Z_2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{26}$$

V_{112}.

The bases W_{ijk} are the same:

$$W_{002} = q^{\tau} \qquad W_{011} = q^{2\tau} \qquad W_{101} = q^{2\tau} \qquad W_{110} = q^{\tau} \tag{27}$$

The products $(nx_\ell, ny_\ell, nz_\ell, n\alpha_{ijk})$ are

$$nx_0 = 1 \tag{28}$$

$$nx_1 = 1 \tag{29}$$

$$ny_0 = 1 \tag{30}$$

$$ny_1 = W_{011}^3 W_{101}^{-3} = 1 \tag{31}$$

$$nz_0 = W_{011}^{-3} W_{101}^3 W_{110}^3 = q^{3\tau} \tag{32}$$

$$nz_1 = W_{101}^3 = q^{6\tau} \tag{33}$$

$$nz_2 = W_{002}^3 = q^{3\tau} \tag{34}$$

giving a final result of

$$V_{022} = (1+1)^{1/3}(1+1)^{1/3}(q^{3\tau} + q^{6\tau} + q^{3\tau})^{1/3} \tag{35}$$

$$= 2^{2/3}(q^{3\tau})^{1/3}(2 + q^{3\tau})^{1/3} \qquad \text{—should be } 2^{2/3}q^{\tau}(q^{3\tau}+2)^{1/3} \checkmark \tag{36}$$